J. Fluid Mech. (1971), *vol.* 50, *part* 4, *pp.* 801–814
Printed in Great Britain

Numerical solutions for steady symmetric viscous flow past a parabolic cylinder in a uniform stream

By S. C. R. DENNIS AND J. D. WALSH

Department of Applied Mathematics, University of Western Ontario, London, Canada

(Received 22 October 1970 and in revised form 29 June 1971)

Numerical solutions are presented for steady two-dimensional symmetric flow past a parabolic cylinder in a uniform stream parallel to its axis. The solutions cover the range $R = 0{\cdot}25$ to ∞, where R is the Reynolds number based on the nose radius of the cylinder. For large R, the calculated skin friction near the nose of the cylinder is compared with known theoretical results obtained from second-order boundary-layer theory. Some discrepancy is found to exist between the present calculations and the second-order theory. For small R, it is possible to obtain a reasonably consistent check with a recent theoretical prediction for the limit of the skin friction near the nose of the cylinder as $R \to 0$.

1. Introduction

One of the main interests in the problem of steady flow past a parabolic cylinder in a uniform stream is that it provides a relatively simple application both of first-order boundary-layer theory and of the calculation of higher order corrections to this theory. Van Dyke (1962*a*, *b*) has considered the general theory in the case of flow past leading edges. For a parabolic cylinder there is no singularity at the nose, as in the case of the semi-infinite flat plate, and the flow proceeds without separation from stagnation point flow at the nose to Blasius flow farther downstream. For this problem Van Dyke (1964*a*) has shown that all the second-order effects can be calculated completely. The result is that the correction of order $R^{-\frac{1}{2}}$ to boundary-layer theory, where R is the Reynolds number based on the nose radius of the cylinder, can be calculated explicitly.

Some check on the theory has been attempted by Wang (1965) for the case of the parabolic cylinder by utilizing the semi-analytical method of series truncation developed by Van Dyke (1964*b*, 1965). However, the published results of Wang's calculations for the skin friction near the nose of the cylinder (Davis 1967) are not in significantly good agreement with Van Dyke's (1964*a*) result. A check on the second-order theory by means of numerical solutions of the Navier–Stokes equations would seem to be appropriate and this is one of the main objectives of the present paper. Numerical solutions are obtained using two-dimensional finite-difference approximations to the partial differential equations for the stream function and vorticity. The Reynolds number range considered is $R = 0{\cdot}25$ to $R = \infty$. From the results it is possible to obtain an estimate of the second-order correction to the skin friction for large R. This estimate indicates some discrepancy with the theoretical result of Van Dyke (1964*a*). There is also a substantial difference between the results of the present calculations and those of Wang (1965) for the lower Reynolds numbers. Good agreement is obtained, however, with recent results of Davis (1972).

Van Dyke (private communication) has recently shown that the limit of the skin friction near the nose of a parabolic cylinder as $R \to 0$ can be related to the skin friction near the leading edge of a semi-infinite flat plate and has calculated an estimate of this limit by utilizing recent results of van de Vooren & Dijkstra (1970) and Yoshizawa (1970) for the flat plate. By assuming the limit given by Van Dyke, it is possible to obtain a reasonable check on the present calculations for small R.

2. Basic equations and boundary conditions

As in the calculations of Wang, parabolic co-ordinates are used since these have been shown (Kaplun 1954) to be optimal for the boundary-layer problem. If (x, y) are dimensionless Cartesian co-ordinates which are related to the corresponding dimensional co-ordinates (x^*, y^*) by the equations $x^* = Lx$, $y^* = Ly$,

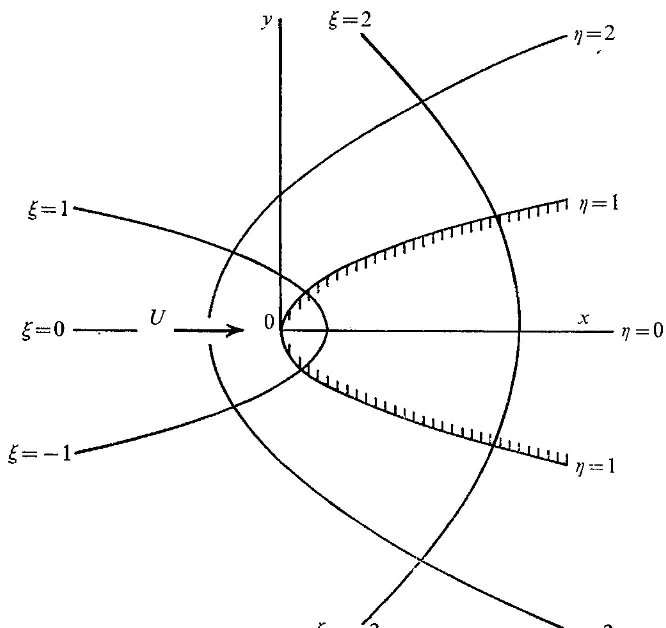

FIGURE 1. Parabolic co-ordinate system.

where L is the nose radius of the parabola, the dimensionless profile of the cylinder is taken as $y^2 = 2x$. Parabolic co-ordinates (ξ, η) are introduced by the transformation

$$x - \tfrac{1}{2} + iy = \tfrac{1}{2}(\xi + i\eta)^2. \tag{1}$$

The co-ordinates are shown in figure 1. The cylinder is situated at $\eta = 1$, and at any point on its surface we have $x = \frac{1}{2}\xi^2$. If the flow at large distances is a uniform stream U parallel to the positive x direction, we shall suppose that the velocity components have been non-dimensionalized with regard to U. Thus if ψ^* and ζ^* are the usual dimensional stream function and vorticity, respectively, the dimensionless functions ψ and ζ are introduced by the equations

$$\psi^* = UL\psi, \quad \zeta^* = -U\zeta/L. \tag{2}$$

In the parabolic co-ordinate system, the governing equations derived from the Navier–Stokes equations are then

$$\frac{\partial^2\zeta}{\partial\xi^2}+\frac{\partial^2\zeta}{\partial\eta^2} = R\left(\frac{\partial\psi}{\partial\eta}\frac{\partial\zeta}{\partial\xi}-\frac{\partial\psi}{\partial\xi}\frac{\partial\zeta}{\partial\eta}\right), \tag{3}$$

$$\frac{\partial^2\psi}{\partial\xi^2}+\frac{\partial^2\psi}{\partial\eta^2} = (\xi^2+\eta^2)\,\zeta. \tag{4}$$

Here, $R = UL/\nu$ is the Reynolds number. Finally, it is convenient to introduce boundary-layer quantities, defined by means of the equations

$$\eta = 1+R^{-\frac{1}{2}}\eta', \quad \psi = R^{-\frac{1}{2}}\psi', \quad \zeta = R^{\frac{1}{2}}\zeta'. \tag{5}$$

If these are substituted in (3) and (4) and the primes are suppressed, for convenience, we obtain

$$R^{-1}\frac{\partial^2\zeta}{\partial\xi^2}+\frac{\partial^2\zeta}{\partial\eta^2} = \frac{\partial\psi}{\partial\eta}\frac{\partial\zeta}{\partial\xi}-\frac{\partial\psi}{\partial\xi}\frac{\partial\zeta}{\partial\eta}, \tag{6}$$

$$R^{-1}\frac{\partial^2\psi}{\partial\xi^2}+\frac{\partial^2\psi}{\partial\eta^2} = \{\xi^2+(1+R^{-\frac{1}{2}}\eta)^2\}\,\zeta. \tag{7}$$

These are the final governing equations. In the rest of the paper it will be assumed that the unprimed quantities in (6) and (7) stand for the primed quantities defined by the equations (5).

The boundary conditions at the surface of the cylinder are

$$\psi = \partial\psi/\partial\eta = 0 \quad \text{when} \quad \eta = 0. \tag{8}$$

The condition that the flow reduces to a uniform stream parallel to the positive x direction at large distances gives

$$\partial\psi/\partial\eta \sim \xi \quad \text{as} \quad \eta \to \infty. \tag{9}$$

As a consequence of (9) it also follows that

$$\zeta \to 0 \quad \text{as} \quad \eta \to \infty. \tag{10}$$

The flow is assumed to be symmetrical about the axis of x and thus

$$\psi = \zeta = 0 \quad \text{when} \quad \xi = 0. \tag{11}$$

For the boundary-layer problem defined by putting $R = \infty$ in (6) and (7), the above conditions are sufficient. The initial conditions are given by (11). The boundary conditions are given by (8) and (9), and (10) may be utilized as an additional condition, if required.

For any finite value of R, equations (6) and (7) are both elliptic and conditions which are valid for large ξ must be specified. The assumption is that the flow for large ξ is governed by the Blasius boundary-layer solution, which implies that

$$\psi(\xi,\eta) \sim \xi f(\eta) \quad \text{as} \quad \xi \to \infty, \tag{12}$$

where $f(\eta)$ satisfies the equation

$$f''' + ff'' = 0,$$

with $$f(0) = f'(0) = 0, \quad f'(\infty) = 1.$$

Here, primes denote differentiation with regard to η. The equation for $f(\eta)$ is the Blasius equation. It can be deduced that $f(\eta)$ must satisfy this equation if (6) and (7) are to be satisfied asymptotically by (12). In the numerical method of solution to be described, in which (6) and (7) are solved as a simultaneous problem, a condition for $\zeta(\xi, \eta)$ as $\xi \to \infty$ is also required. The leading term for ζ corresponding to the asymptotic Blasius flow is

$$\zeta(\xi, \eta) \sim \xi^{-1} f''(\eta) \quad \text{as} \quad \xi \to \infty.$$

In practice it has been found to be more satisfactory to assume the expression (12) and then determine an expression for ζ from (7) by satisfying this equation exactly. This yields

$$\zeta(\xi, \eta) \sim \frac{\xi f''(\eta)}{\xi^2 + (1 + R^{-\frac{1}{2}}\eta)^2} \quad \text{as} \quad \xi \to \infty. \tag{13}$$

The expression (12) is the leading term of the boundary-layer type expansion, valid for large ξ, considered by Van Dyke (1964*a*) and by others (Imai 1957; Stewartson 1957; Goldstein 1960; Murray 1965). Similar expressions to (12) and (13) have been used by Yoshizawa (1970) as boundary conditions for large ξ when considering the case of the semi-infinite flat plate by means of numerical solutions. A more elaborate treatment of the conditions for large ξ is given by van de Vooren & Dijkstra (1970) in their numerical solution of the semi-infinite flat plate problem. In the present work the conditions (12) and (13) are used as boundary conditions for large ξ. In practice they will be applied at some finite value, $\xi = \xi_m$, of ξ as a first approximation and the effect on the numerical results of increasing ξ_m will subsequently be studied. Likewise, the conditions (9) and (10) are first assumed to be valid on some boundary $\eta = \eta_m$ at large enough distance from $\eta = 0$. It is subsequently found that the imposition of the boundary at the finite distance $\xi = \xi_m$ is satisfactory, but that the approximation of the condition as $\eta \to \infty$ requires further consideration. This will be described later.

Numerical approximations to ψ and ζ are first obtained at the points of a rectangular grid within the finite rectangle bounded by the lines $\xi = 0$, $\xi = \xi_m$, $\eta = 0$, $\eta = \eta_m$. Considerable care is necessary in formulating the numerical approach. For finite values of R, (6) and (7) give rise to an elliptic boundary-value problem. If $R = \infty$, (6) is a parabolic equation for the function ζ and the boundary-layer problem of solving (6) and (7) is a step-by-step integration in the ξ direction, starting from the initial conditions (11). The numerical method must be capable of dealing with both the elliptic and parabolic cases. This is achieved by using a suitable method of approximating (6) in terms of finite differences. This method will be discussed in the next section.

3. Numerical method of solution

Although some of the calculations were carried out using unequal grid sizes in the ξ and η directions, it is simpler to describe the method using equal grid sizes in both directions. The extension to unequal grid sizes is obvious. The system of numbering a typical set of points on a square grid of side h is indicated in figure 2. Central differences are used to approximate the derivatives in (7) at the typical point O, which yields

$$R^{-1}\psi_1+\psi_2+R^{-1}\psi_3+\psi_4-2(1+R^{-1})\psi_0-h^2\chi_0\zeta_0=0, \qquad (14)$$

where $$\chi=\xi^2+(1+R^{-\frac{1}{2}}\eta)^2,$$

and the subscripts denote local values of the appropriate quantities. Equation (14) must be satisfied at every internal grid point, subject to the boundary conditions that $\psi=0$ on both $\xi=0$ and $\eta=0$ and that ψ is given by (12) on $\xi=\xi_m$.

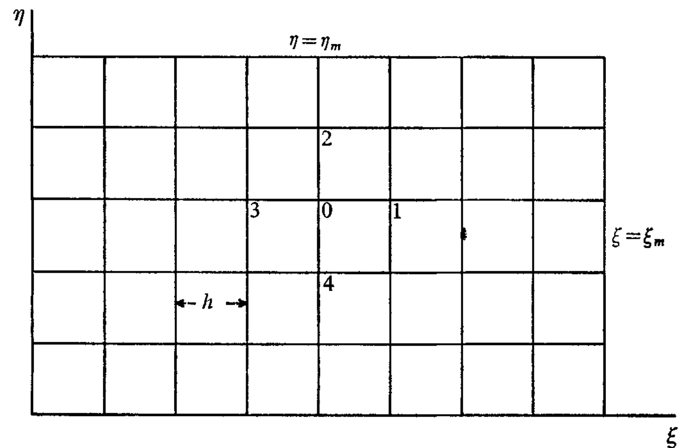

FIGURE 2. Region of integration and grid structure.

Equation (14) also holds for all grid points on $\eta=\eta_m$ in conjunction with the condition (9) which, when the derivative is expressed in central differences, gives the approximation
$$\psi_2=\psi_4+2h\xi_0.$$

Thus (14) becomes, for points on $\eta=\eta_m$,

$$R^{-1}(\psi_1+\psi_3)+2\psi_4-2(1+R^{-1})\psi_0+2h\xi_0=0, \qquad (15)$$

since $\zeta_0=0$ on $\eta=\eta_m$, as an approximation to (10). For a given value of ζ_0 at each internal grid point, the solution of the set of equations defined by (14) and (15) is a standard problem which presents no difficulties.

The problem of approximating (6) in a manner which is satisfactory for all R must now be considered. It is convenient to write

$$2\lambda=\partial\psi/\partial\eta, \quad 2\mu=\partial\psi/\partial\xi. \qquad (16)$$

The functions λ and μ can then be calculated approximately from grid values of ψ. For the moment we shall defer consideration of the appropriate finite-difference formulae to be used for this calculation. Several ways of approximating (6) by finite differences are possible. If all derivatives of ζ in (6) are approximated by central differences at the typical point O, the approximation

$$(R^{-1}-h\lambda_0)\zeta_1+(1+h\mu_0)\zeta_2+(R^{-1}+h\lambda_0)\zeta_3+(1-h\mu_0)\zeta_4-2(1+R^{-1})\zeta_0=0 \quad (17)$$

is obtained. This must be satified at all internal grid points. The associated boundary conditions are that $\zeta=0$ on $\xi=0$, and ζ may also be taken as zero on $\eta=\eta_m$, as an approximation to (10). On $\xi=\xi_m$, ζ is assumed to be given by (13). Finally, the second condition in (8) may be used to give a condition for ζ on $\eta=0$. When the typical point O of figure 2 is on $\eta=0$, the central-difference approximation to the condition $(\partial\psi/\partial\eta)_0=0$ gives $\psi_2=\psi_4$. If this is substituted in (14), with $\psi_0=\psi_1=\psi_3=0$, we obtain

$$\zeta_0=2\psi_2/\{(1+\xi_0^2)h^2\} \quad (18)$$

and thus boundary values of ζ on $\eta=0$ are calculated from values of ψ on $\eta=h$. With these boundary conditions and values of λ_0 and μ_0 in (17) calculated from the solution for ψ, an approximation to ζ can be found at each grid point.

A numerical solution for a given value of R is obtained by solving the sets of equations (14), (15) and (17), subject to the stated boundary conditions, by an overall iterative procedure. This procedure is repeated until ψ and ζ have converged to limits, within an acceptable tolerance, at every internal grid point and also at every boundary grid point at which they are not known. One of the acceleration devices used in the overall procedure is that of successive over-relaxation. This is utilized in solving (14) and (15) for ψ, and also in solving (17) for ζ. Successive over-relaxation has been generally described by Varga (1962) and by Wachspress (1966). It can be divergent if the matrix associated with the equations is not diagonally dominant in the sense defined by Varga (1962, p. 23) and, indeed, strict diagonal dominance is a sufficient condition for the convergence of the point Jacobi and Gauss–Seidel iterative procedures associated with a linear system (Varga 1962, p. 73).

The equations (17), looked on as a linear system in ζ, can be written as

$$\sum_{n=1}^{4} c_n\zeta_n-c_0\zeta_0=0 \quad (19)$$

at every internal grid point of the rectangle. The condition for diagonal dominance is that

$$\Sigma|c_n|\leqslant c_0 \quad (20)$$

at every internal grid point, where the summation applies to the four c_n associated with the particular grid point in general, but at points adjacent to the boundaries the one c_n which multiplies the boundary value is omitted. Since no reversed flow takes place, the functions λ and μ are always positive within the rectangle. It is obvious that (20) is satisfied provided that both of the conditions

$$\lambda_0 h\leqslant R^{-1}, \quad \mu_0 h\leqslant 1 \quad (21)$$

are satisfied at every grid point. When they are satisfied, the strict inequality in (20) applies at all points adjacent to the boundary and the diagonal dominance is therefore strict.

In the numerical calculations it was found to be relatively easy to choose h to satisfy the second of (21) within the rectangle, but not the first. The overall iterative procedure failed to converge for quite small R, not much greater than $R = 1$, although solutions for $R = 0{\cdot}25$, $0{\cdot}5$ could be obtained satisfactorily. In order to obtain solutions for higher values of R, a method similar to that proposed by Spalding (1967) and later by Greenspan (1968) in studies of the Navier–Stokes equations was used, although some extension of these ideas is made, following the method suggested by Dennis & Chang (1969). The basic principle is to write the usual central-difference approximation to the term $\partial\zeta/\partial\xi$ in (6) as a corrected backward difference, thus

$$h(\partial\zeta/\partial\xi)_0 = \zeta_0 - \zeta_3 + C_0, \tag{22}$$

where

$$C_0 = \tfrac{1}{2}(\zeta_1 - 2\zeta_0 + \zeta_3). \tag{23}$$

The finite-difference equations (17) can then be expressed in the alternative form

$$R^{-1}\zeta_1 + (1 + h\mu_0)\,\zeta_2 + (R^{-1} + 2h\lambda_0)\,\zeta_3 + (1 - h\mu_0)\,\zeta_4 - 2(1 + R^{-1} + h\lambda_0)\,\zeta_0 = 2h\lambda_0 C_0. \tag{24}$$

A valid approximation to (6) is obtained by neglecting C_0 in (24), since $C_0 = O(h^2)$. This yields the finite-difference equations

$$R^{-1}\zeta_1 + (1 + h\mu_0)\,\zeta_2 + (R^{-1} + 2h\lambda_0)\,\zeta_3 + (1 - h\mu_0)\,\zeta_4 - 2(1 + R^{-1} + h\lambda_0)\,\zeta_0 = 0, \tag{25}$$

and the associated matrix is diagonally dominant provided that only the second of (21) is satisfied. This condition does not depend strongly on R and is easily satisfied over the rectangle. Thus the overall iterative procedure of solving (25) with (14) and (15) was found to converge for all R; in particular, it was possible to solve the case $R = \infty$ by this method.

When $R = \infty$, the terms in R^{-1} in (6) and (7) are zero. Equation (6) is parabolic and the whole integration can be performed as a step-by-step integration in the ξ direction, since there is no forward influence in the ξ direction in (25), (14) and (15) in this case, and the numerical solution may be completed along a line of constant ξ, starting with $\xi = h$, before proceeding to the next. To preserve the truly step-by-step nature of the integration it is necessary to express λ in (16) by a backward-difference approximation. Thus we may take

$$2h\lambda_0 = \psi_0 - \psi_4, \quad 4h\mu_0 = \psi_1 - \psi_3 \tag{26}$$

as numerical approximations to (16). It is easily verified that with this scheme the numerical solution along $\xi = h$, subject to the initial conditions (11), is a numerical formulation of the exact Hiemenz stagnation point solution (Schlichting 1960, p. 78), i.e. it gives the numerical functions which multiply the power of ξ in the Blasius series for ψ and ζ when expressed in the parabolic co-ordinate system (Van Dyke 1964*a*). Once the solution on $\xi = h$ has been found we can proceed to $\xi = 2h$. The only backward influence is through the term ψ_4 in (26) and the term ζ_4 in (25), which are both known.

The boundary-layer case $R = \infty$ suggests how the overall iterations should proceed in the general case. The rectangle is swept along lines of constant ξ, starting from $\xi = h$ and finishing at $\xi = \xi_m - h$. The iterations performed by Yoshizawa (1970) in the case of the semi-infinite flat plate employ sweeps along the lines of constant η but, other than this, the procedure adopted here is similar. Some of the devices for accelerating convergence which are described in numerical studies (e.g. by Kawaguti 1961; Burggraf 1966; Takami & Keller 1969) have been employed. In particular, the calculation of boundary values of ζ on $\eta = 0$ during the course of the overall iterations is not done directly from (18) but by an averaging process. If an approximation $\zeta^{(m)}$ throughout the region gives rise to an approximation $\psi^{(m)}$ by solving (14) and (15) with $\zeta = \zeta^{(m)}$, the boundary values of ζ used in the next iteration are calculated from

$$\zeta^{(m+1)}(\xi, 0) = \frac{2\omega\psi^{(m)}(\xi, h)}{h^2(1+\xi^2)} + (1-\omega)\,\zeta^{(m)}(\xi, 0). \tag{27}$$

Here, ω is a parameter such that $0 < \omega \leqslant 1$. The case $\omega = 1$ corresponds to direct calculation of new boundary values from (18). If this causes divergence of the iterations, ω is reduced in magnitude until convergence is obtained. Finally, in the general case of finite R it is quite convenient to employ the central-difference formula

$$4h\lambda_0 = \psi_2 - \psi_4 \tag{28}$$

instead of the first of (26). Actually this was also employed in the case $R = \infty$, which was simply treated as another case of the general procedure for finite R and solved as if it were an elliptic problem.

Some account must also be taken of the fact that (17) are correct to full central-difference accuracy, whereas (25) employ only backward-difference accuracy in approximating $\partial\zeta/\partial\xi$ and therefore lead to a less accurate solution. The procedure described by Dennis & Chang (1969) is adopted. From a first approximation which satisfies (25), the correction C_0 is calculated at each point. The iterations are continued using (24) with C_0 substituted. Every so often (say, every 20 or 30 complete iterations) C_0 is re-calculated from the current solution and the whole procedure repeated until convergence. This is similar to the method of Fox (1947). It was found to be convergent in every case. In this way, approximations which satisfy (17) were obtained for all R, whereas iterative methods applied directly to (17) certainly diverged for $R > 1$.

Numerical solutions were obtained for a range of values of R from $R = 0{\cdot}25$ to $R = \infty$. The same grids were used for all values of R but, as has been stated, unequal grid sizes in the ξ and η directions were used in some of the solutions. If H denotes the grid size in the η direction, solutions were first obtained taking $h = \frac{1}{5}$, $H = \frac{1}{10}\sqrt{2}$. The grid size H was taken to be consistent with that used (Schlichting 1960, p. 121) in calculating the Blasius function $f(\eta)$ which is used in (12) and (13) (Schlichting's variable η is $\sqrt{2}$ times that used here). A good check that H has been properly chosen is given by the fact that the solution of (14), (15) and (25) along the line $\xi = h$ in the boundary-layer case $R = \infty$ gives a very good comparison with the Hiemenz stagnation point solution. A second solution was obtained for each value of R, taking $h = \frac{1}{10}$ and keeping $H = \frac{1}{10}\sqrt{2}$. The change in the solution due to this change in grid size was hardly more than 1 % for any value of R.

The outer boundary $\eta = \eta_m$ was taken sufficiently far from $\eta = 0$ for the function $f'(\eta)$ to have assumed its free-stream value to five decimal places, and the position of the boundary $\xi = \xi_m$ was taken to be consistent with that taken by Yoshizawa (1970) in the case of flow past a semi-infinite flat plate. Finally, in the overall iterative procedure, convergence was decided by the test $M < \epsilon$, where

$$M = \max\left|\frac{\zeta^{(r+1)}(\xi,0)-\zeta^{(r)}(\xi,0)}{\zeta^{(r+1)}(\xi,0)}\right|$$

for all ξ in the range $0 < \xi < \xi_m$, and where r, $r+1$ are two successive iterates. The value of the parameter ϵ was varied to suit different values of R but on the whole ϵ was always less than 10^{-4}.

The effect of varying the positions of both of the boundaries $\xi = \xi_m$ and $\eta = \eta_m$ was studied. The position of each boundary was varied independently. It was found that for sufficiently large ξ_m (> 20), the effect of increasing ξ_m was quite negligible to the order of accuracy considered. The effect of increasing η_m was much more significant. It was found that even after considerable increase in η_m, the calculated skin friction at the nose of the cylinder was rather higher (the maximum discrepancy was of the order of 2 %) than that computed by Davis (1972). The probable reason is that whereas the vorticity ζ decays exponentially to zero as $\eta \to \infty$, the decay to its ultimate value as $\eta \to \infty$ of the perturbation stream function Ψ defined by the equation

$$\psi = \xi\eta + \Psi \tag{29}$$

is algebraic (see, for example, the second-order theory of Van Dyke (1964*a*)). The method of satisfying the boundary condition for ψ as $\eta \to \infty$ was therefore modified according to the following procedure.

If η_m is large enough we can put $\zeta = 0$ in (7) for $\eta > \eta_m$. We now substitute in the resulting equation for ψ from (29) and make the transformation $\eta = 1/z$. The equation for $\Psi(\xi, z)$ in the region $0 \leqslant z \leqslant z_m$, where $z_m = 1/\eta_m$, is

$$z^4\frac{\partial^2\Psi}{\partial z^2} + 2z^3\frac{\partial\Psi}{\partial z} + R^{-1}\frac{\partial^2\Psi}{\partial\xi^2} = 0. \tag{30}$$

The work of Van Dyke (1964*a*) suggests that it is reasonable to expand Ψ in integer powers of z, if z is small enough. We therefore take a grid size $k = \frac{1}{2}z_m$ in the variable z, so that $z = 2k$ corresponds to $\eta = \eta_m$, and replace the derivatives in (30) by central-difference expressions at grid points along the line $z = k$. The approximation to (30) along this line at a grid point for which $\xi = \xi_0$ is then

$$2\Psi(\xi_0, 2k) + \gamma\{\Psi(\xi_0 + h, k) + \Psi(\xi_0 - h, k)\} - 2(1+\gamma)\Psi(\xi_0, k) = 0, \tag{31}$$

where $\gamma = 1/(Rh^2k^2)$. It may be noticed that the term involving $\Psi(\xi_0, 0)$ has cancelled, so that the boundary condition for Ψ at $z = 0$ does not have to be specified. The term in $\Psi(\xi_0, 2k)$ in (31) can be linked to the value of ψ at the same point $\xi = \xi_0$, $\eta = \eta_m$ by means of (29). This value of ψ is that denoted by ψ_0 in equation (15) when the typical point O of figure 2 is on the line $\eta = \eta_m$.

The effect of the grid values $\Psi(\xi_0, k)$ on the solution for $\eta \leqslant \eta_m$ is introduced by replacing the boundary condition (9) by the condition, obtained from (29), that

$$\partial\psi/\partial\eta = \xi - z^2\,\partial\Psi/\partial z, \quad \text{when} \quad \eta = \eta_m. \tag{32}$$

The term $\partial\Psi/\partial z$ in (32) is now replaced by a simple backward difference of Ψ with regard to the z variable at $z = z_m$, and the resulting approximation to the right side of (32) is now used to replace the right side of (9) in deriving the equation (15) for the function ψ along the grid line $\eta = \eta_m$. Thus, in terms of the grid structure in figure 2, the approximation to (32) when the typical point O is on $\eta = \eta_m$ gives

$$\psi_2 = \psi_4 + 2h\xi_0 - 8hk\{\Psi(\xi_0, 2k) - \Psi(\xi_0, k)\}. \tag{33}$$

If we eliminate $\Psi(\xi_0, 2k)$ from (33) using (29) and then eliminate ψ_2 from (14), as before, the modified form of (15) becomes

$$R^{-1}(\psi_1 + \psi_3) + 2\psi_4 - 2(1 + R^{-1} + 4hk)\,\psi_0 + 8hk\,\Psi(\xi_0, k) + 2h\xi_0(1 + 4k\eta_0) = 0. \tag{34}$$

The two sets of equations (31) and (34), with $\Psi(\xi_0, 2k)$ in (31) related to ψ_0 in (34) by means of (29), serve to replace the single set (15) as a representation of the boundary condition as $\eta \to \infty$. With this representation the problem is solved, effectively, as a two-region problem with ψ and $\partial\psi/\partial\eta$ continuous across the interface $\eta = \eta_m$. The computation extends over the complete range $\eta = 0$ to $z = 0$. The equations are solved by the following iterative procedure. The term involving $\Psi(\xi_0, k)$ is held fixed in (34) and these equations are solved iteratively in conjunction with the appropriate equations in the inner region $\eta < \eta_m$ for a definite number of iterative sweeps of the whole field $\eta \leqslant \eta_m$. This is carried out in the manner already described. Values of $\Psi(\xi_0, 2k)$ in (31) are then calculated from (29) and held fixed while (31) are solved iteratively along the line $z = k$ until convergence is achieved to some prescribed accuracy. The iterations in the region $\eta \leqslant \eta_m$ are then continued with the new $\Psi(\xi_0, k)$ held fixed in (34), and so on until ultimate convergence of the whole sequence of operations is obtained. The final calculations according to this scheme were carried out with $h = \frac{1}{10}$, $H = \frac{1}{10}\sqrt{2}$ and $\eta_m = 5\sqrt{2}$. This latter value gives $k = 0{\cdot}1/\sqrt{2}$. It will be seen below that the results calculated using this modified scheme are in excellent agreement with those obtained by Davis (1972).

4. Calculated results

The dimensionless skin friction is $c_f = \tau_0/(\frac{1}{2}\rho U^2)$, where τ_0 is the local shearing stress, and hence $c_f = 2\zeta_0/R^{\frac{1}{2}}$. If the local Reynolds number $R_x = Ux^*/\nu$ is introduced, it is found that

$$R_x^{\frac{1}{2}} c_f = \sqrt{2}\,\xi\zeta_0. \tag{35}$$

In all these expressions, the subscript 0 refers to values on the cylinder. Near the nose of the cylinder ζ_0 has the form $\zeta_0 \sim A\xi$, where $A = (\partial\zeta/\partial\xi)_0$, and it follows that A is the value of $(R/8x)^{\frac{1}{2}}\,c_f$ at the nose. This quantity has been calculated by Wang (1965) and the results for the first truncation of the series truncation method used by Wang are given by Davis (1967). Values of A calculated from the present solutions are given for the range of R considered in table 1. These results are compared with Wang's results in figure 3, which also shows the results obtained from the second-order boundary-layer theory of Van Dyke (1964*a*). There appears to be a small, but significant, difference in the present results from those of Wang, and the comparison with Van Dyke's theoretical result is not as precise as may be expected. The comparison with the recent calculations of Davis (1971) is excellent, however. At values of R at which a direct numerical comparison is possible ($R = 1, 10, 100, 10^3$) the difference is not anywhere more than about 0·3 %.

R	$A = (\partial\zeta/\partial\xi)_0$	R	$A = (\partial\zeta/\partial\xi)_0$
0·25	0·576	10^3	1·149
0·5	0·598	5×10^3	1·196
1	0·627	$2{\cdot}5 \times 10^4$	1·218
5	0·732	5×10^4	1·222
10	0·793	∞	1·232
100	1·007	—	—

TABLE 1. Values of $(R/8x)^{\frac{1}{2}} c_f$ at the nose of a parabolic cylinder

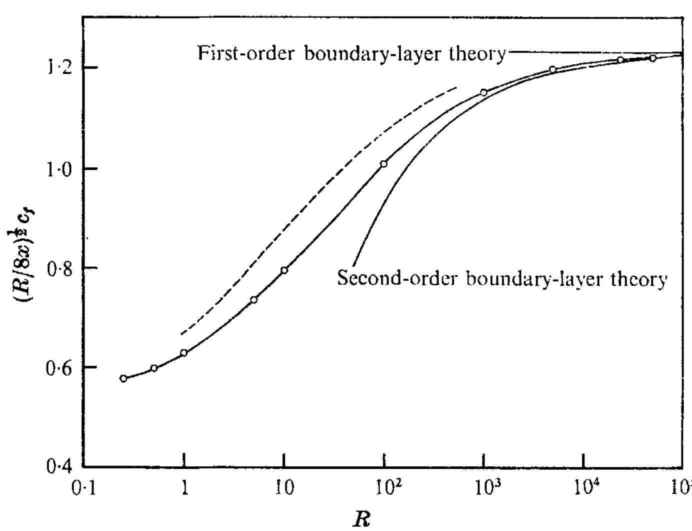

FIGURE 3. Values of $(R/8x)^{\frac{1}{2}} c_f$ at the nose of the cylinder, $x = 0$; –○–, Dennis & Walsh; ----, Wang (1965).

Van Dyke's second-order theory gives the skin friction in the neighbourhood of the nose as

$$\tfrac{1}{2}R^{\frac{1}{2}}c_f = (1{\cdot}232588 - 3{\cdot}03R^{-\frac{1}{2}})(s/L) + O(s^3, R^{-1}), \tag{36}$$

where s is the dimensional distance along the cylinder surface, measured from the nose. For high R, the coefficient of s/L on the right side should correspond approximately with the values of A in table 1. The comparison at $R = \infty$ is satisfactory, but the present values for high R are higher than those given by (36). We can attempt an estimate of the coefficient of $R^{-\frac{1}{2}}$ from the present results by writing

$$A \sim a + bR^{-\frac{1}{2}}. \tag{37}$$

We take the value $a = 1\cdot2326$ consistent with the exact Hiemenz solution and calculate b from (37) at $R = 2\cdot5\times10^4$ and $R = 5\times10^4$ using accurately calculated values of $A = 1\cdot2176$ and $A = 1\cdot2222$, respectively, at these values of R. We then get the two estimates, respectively, of $b = -2\cdot37$ and $b = -2\cdot33$. An extrapolation to $R = \infty$ in $R^{-\frac{1}{2}}$ then gives the result $b = -2\cdot21$. This type of estimation is highly conjectural, however, because it must be noted that we obtain the calculated value $a = 1\cdot2320$ at $R = \infty$ from the present results, which is lower than the exact value. A similar error could exist in the results at $R = 2\cdot5\times10^4$ and $R = 5\times10^4$ but, even so, the estimate of b could hardly be in error by as much as the result (36) indicates.

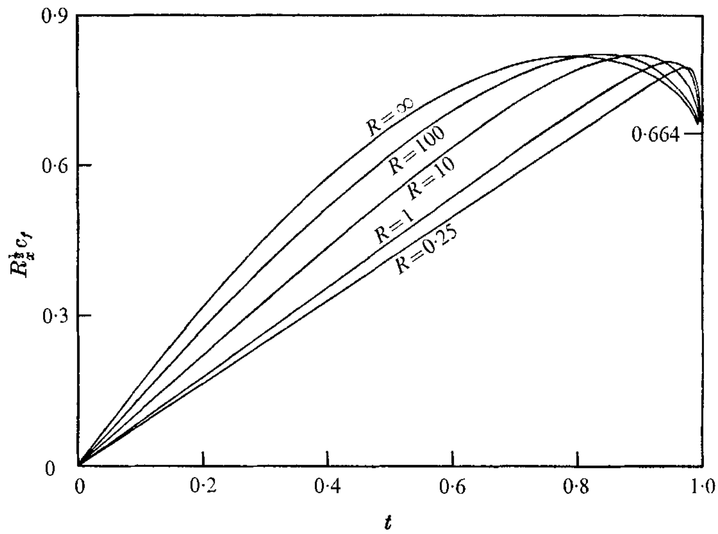

FIGURE 4. $R_x^{\frac{1}{2}}c_f$ as a function of $t = \xi^2/(1+\xi^2)$ over the cylinder surface.

The skin friction over the surface of the cylinder for various Reynolds numbers is plotted against the co-ordinate $t = \xi^2/(1+\xi^2)$ used by Van Dyke (1964*a*) in figure 4. The curve for $R = \infty$ is compared with the boundary-layer calculations of Smith & Clutter (1963) in figure 5. The present values are somewhat higher than those of Smith & Clutter as ξ increases; for example, at $\xi = 2$ ($t = 0\cdot8$) we obtain the value $R_x^{\frac{1}{2}}c_f = 0\cdot8200$ compared with Smith & Clutter's value $R_x^{\frac{1}{2}}c_f = 0\cdot8152$. Van Dyke (1964*a*) has noted that, for large ξ, Smith & Clutter's results are some 0·2 to 0·3 % lower than those of Fannelop (unpublished). In any event, there is good mutual agreement between all the results near the nose of the cylinder.

At the low end of the Reynolds number scale, Van Dyke (private communication) has shown that the limit of the skin friction near the nose of the cylinder as $R \to 0$ can be related to the skin friction at the leading edge of a semi-infinite flat plate. By using the recent estimates of the leading edge skin friction given by van de Vooren & Dijkstra (1970) and Yoshizawa (1970) for the semi-infinite flat plate, Van Dyke obtains the result

$$\tfrac{1}{2}R^{\frac{1}{2}}c_f = \{0\cdot532 + O(R^{\frac{1}{2}})\}(s/L) + O(s^3/L^3) \qquad (38)$$

for the skin friction near the nose of the cylinder as $R \to 0$. If we assume

$$\tfrac{1}{2}R^{\frac{1}{2}}c_f \sim (0{\cdot}532 + BR^{\frac{1}{2}})(s/L) \tag{39}$$

as R and s both tend to zero, and then estimate B from the result for $R = 0{\cdot}25$ in table 1, we obtain $B = 0{\cdot}088$. The formula (39) then fits the results for $R = 0{\cdot}5$ and 1 in table 1 to about 1%, giving a consistent check. Van Dyke (private communication) has given a further comparison with the present numerical results.

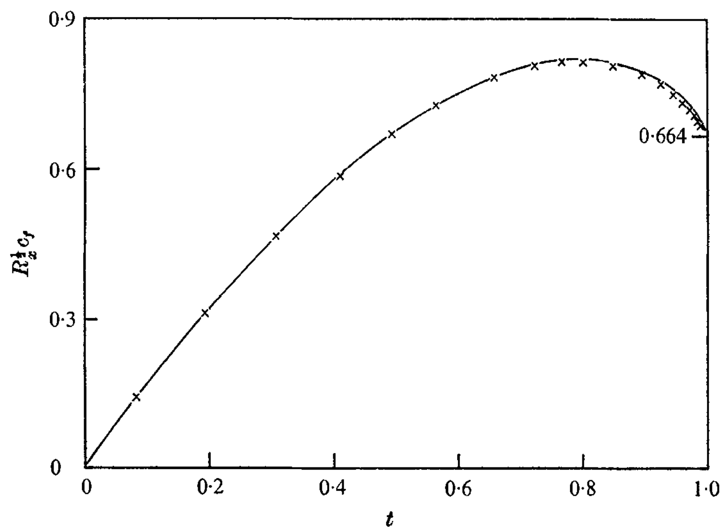

FIGURE 5. $R_x^{\frac{1}{2}}c_f$ as a function of $t = \xi^2/(1+\xi^2)$ over the cylinder surface for the case $R = \infty$: ——, Dennis & Walsh; ×, Smith & Clutter (1963).

The present work has been carried out with the assistance of a grant from the National Research Council of Canada. A preliminary account of the results (obtained before the modified treatment of the conditions as $\eta \to \infty$ was applied) was presented at the 12th British Theoretical Mechanics Colloquium held at the University of East Anglia, Norwich, England from 23–26 March 1970. The authors acknowledge helpful comments and criticisms by R. T. Davis and M. Van Dyke.

REFERENCES

BURGGRAF, O. R. 1966 *J. Fluid Mech.* **24**, 113.

DAVIS, R. T. 1967 *J. Fluid Mech.* **27**, 691.

DAVIS, R. T. 1972 *J. Fluid Mech.* (to appear).

DENNIS, S. C. R. & CHANG, G. Z. 1969 *Phys. Fluids Suppl.* II, **12**, II–88.

FOX, L. 1947 *Proc. Roy. Soc.* A **190**, 31.

GOLDSTEIN, S. 1960 *Lectures on Fluid Mechanics*. Interscience.

GREENSPAN, D. 1968 In *Lectures on the Numerical Solution of Linear, Singular and Nonlinear Differential Equations*, chapter 10. Prentice Hall.

IMAI, I. 1957 *J. Aeron. Sci.* **24**, 155.

KAPLUN, S. 1954 *Z. angew. Math. Phys.* **5**, 111.

KAWAGUTI, M. 1961 *J. Phys. Soc. Japan*, **16**, 2307.

MURRAY, J. D. 1965 *J. Fluid Mech.* **21**, 337.

SCHLICHTING, H. 1960 *Boundary Layer Theory*. McGraw-Hill.

SMITH, A. M. O. & CLUTTER, D. W. 1963 *Douglas Aircraft Corp. Engng. Papers*, 1525, 1530. See also *A.I.A.A. J.* **1**, 2062.

SPALDING, D. B. 1967 *Numerical Methods for Viscous Flows*. AGARD Conference Proceedings no. 60.

STEWARTSON, K. 1957 *J. Math. Phys.* **36**, 173.

TAKAMI, H. & KELLER, H. B. 1969 *Phys. Fluids Suppl.* II, **12**, 11–51.

VAN DYKE, M. 1962*a* *J. Fluid Mech.* **14**, 161.

VAN DYKE, M. 1962*b* *J. Fluid Mech.* **14**, 481.

VAN DYKE, M. 1964*a* *J. Fluid Mech.* **19**, 145.

VAN DYKE, M. 1964*b* *Proc. Int. Hypersonics Symp.* 1964. Cornell University Press.

VAN DYKE, M. 1965 VII *Symposium on Advanced Problems and Methods in Fluid Dynamics*. Poland, Jarata 1965.

VARGA, R. S. 1962 *Matrix Iterative Analysis*. Prentice-Hall.

VAN DE VOOREN, A. I. & DIJKSTRA, D. 1970 *J. Engng. Math.* **4**, 9.

WACHSPRESS, E. L. 1966 *Iterative Solution of Elliptic Systems and Applications to the Neutron Diffusion Equations of Reactor Physics*. Prentice-Hall.

WANG, R. S. 1965 Virginia Polytechnic Institute, M.S. thesis.

YOSHIZAWA, A. 1970 *J. Phys. Soc. Japan*, **28**, 776.